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Algebraic Approximation for the Elliptic Integral of the First Kind: Application to Pendulum-Like and Duffing-Type Oscillators

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Abstract
The exact period of pendulum-like and Duffing-type oscillators can be derived in terms of a special function called the complete elliptic integral of the first kind, $K(m)$. In order to treat such oscillators in undergraduate physics, mechanics or vibration courses, it is necessary to use accurate solutions that are expressed in terms of elementary functions rather than special functions. In this paper, an approximate algebraic solution to the $K(m)$ function was derived based on fourth-term approximation of the arithmetic-geometric mean (AGM). This approximation was found to compute $K(m)$ to less than $8.0 \times 10^{-4}\%$ relative error for $-10000 \leq m \leq 0.9999$. The AGM approximation of the $K(m)$ function was applied to derive approximate solutions for the periods of pendulum-like and Duffing-type oscillators. In the case of the pendulum-like oscillator, the present approximation gives a maximum relative error of 0.0342% for $0 < A \leq 179.9°$, whereas a maximum relative error of $1.2 \times 10^{-14}\%$ was computed for cubic and cubic-quintic Duffing-type oscillators with positive stiffness constants when $A \rightarrow \infty$.

Keywords: *Pendulum, Duffing oscillator, elliptic integral, arithmetic-geometric mean, strong nonlinear oscillations*

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1. Introduction

The complete elliptic integral of the first kind is a special function that is useful for deriving the periodic solution of nonlinear oscillators. It provides exact solutions to two classes of well-known oscillators, namely: pendulum-like and Duffing-type oscillators. Applications of these oscillators abound in physics, applied mathematics and engineering. Examples of pendulum-like systems include simple pendulum (Big-Alabo, 2020a), compound pendulum (Hinrichsen, 1981), eccentrically-loaded hoop (Cross, 2020), elliptic filters (Lima, 2008), optically-torqued nanorods (Lima, 2008), gravitational lensing in general relativity (Farley, 2018), rigid rod sliding on the smooth walls of a hollow cylinder (Nayfeh and Mook, 1995) and small solid cylinder rolling without slip on the smooth surface of a hollow cylinder (Nayfeh and Mook, 1995). On the other hand, applications of cubic and cubic-quintic Duffing oscillators can be found in the analysis of beams (Sedighi et al, 2013), plates (Israr et al, 2009), snap-through mechanism (Kovacic and Brennan, 2011), quasi-zero stiffness isolator (Kovacic and Brennan, 2011), electronic packaging (Kuang et al, 2019), carbon nanotubes (Sobamowo, 2017), nonlinear wave mechanics (Jordan and Smith, 2007) and propagation of short electromagnetic pulse in nonlinear medium (Belendez et al, 2016). A treatise on Duffing oscillators can be found in Kovacic and Brennan (2011).

Some special integral functions such as gamma and beta functions are introduced in undergraduate courses because their solution can be expressed exactly in terms of elementary functions. For other integral functions such as the error and complementary error functions, solutions are available in the form of tables (Cengel and Turner, 2005) that can be easily read by undergraduate students. However, elliptic integral functions are usually not taught in undergraduate courses because they do not have exact elementary solutions and the available tabulated solutions, which cover a limited input range, are unpopular.

There are three complete elliptic integral functions as follows: the complete elliptic integral of the (a) first kind (b) second kind and (c) third kind. Of these three, the complete elliptic integral of the first kind, i.e. the $K(m)$ function, is the most common and it has application in the derivation of exact and approximate periodic solutions of nonlinear oscillators (Cveticanin, 2018). It can be expressed as an infinite series, but the series solution converges slowly and requires very many terms to obtain the required accuracy for some input values (Big-Alabo, 2020b). Another way to evaluate the $K(m)$ function is numerically. For instance, Mathematica package has a function, EllipticK[·], for numerical evaluation of $K(m)$. The numerical solutions are usually very accurate but are not as attractive for undergraduate courses as the explicit analytical solutions. The connection between the AGM and the $K(m)$ function has been long established and was first published in 1818 by *Carl Friedrich Gauss*. Carvalhaes and Suppes (2008) applied the AGM to obtain an approximate period for the simple pendulum. They derived an explicit formula for the pendulum period using a fourth-term approximation of the AGM solution for the $K(m)$ function. Their results showed that $K(m)$ can be estimated accurately using an approximate AGM solution that is expressible in terms of elementary functions. However, the general accuracy of the fourth-term AGM approximation was not explored outside its application to the simple pendulum, for which $0 \leq m < 1.0$. Furthermore, their expression for the fourth-term AGM approximation is too lengthy (see Equation (A.8) in the appendix) and would require spreadsheet implementation.

In this article, the fourth-term AGM approximation was used to derive a compact algebraic solution for the $K(m)$ function that is expressible in terms of elementary functions. The present approximation of $K(m)$ is algebraically simpler than that of Carvalhaes and Suppes (2008) and can be evaluated with a pocket calculator. The accuracy of the present AGM solution for the $K(m)$ function was investigated for a wide range of inputs and thereafter the AGM solution was used to derive approximate periodic solutions for pendulum-like and Duffing-type oscillators. Comparison of the approximate periodic solutions with corresponding exact periodic solutions was conducted and revealed the accuracy of the present method. Given that the approximate periodic solutions are expressed in terms of elementary functions, the present approach can be used to introduce nonlinear oscillators in undergraduate courses on advanced mechanics or vibration.

2. AGM approximation for $K(m)$

The complete elliptic integral of the first kind is given by the following integral function:

$$K(m) = \int_0^{\pi/2} (1 - m\sin^2\theta)^{-1/2}\, d\theta \qquad (1)$$

For $m \geq 0$, the $K(m)$ function increases gradually for smaller values of m and then rapidly approaches infinity in an asymptotic manner as m approaches unity (see Fig. 1). The region where $m \to 1.0$ is usually difficulty to estimate and the series representation of $K(m)$ would require hundreds to thousands of terms to produce the required accuracy. The implication is that other approximation schemes are needed. The $K(m)$ function can be expressed exactly in terms of the AGM function as shown:

$$K(m) = \frac{\pi}{2M\left(1, \sqrt{1-m}\right)} \qquad (2)$$

where $M\left(1, \sqrt{1-m}\right)$ is the AGM of 1 and $\sqrt{1-m}$, $m = k^2$ is the elliptic parameter and k is the elliptic modulus or eccentricity.

Now, let us consider the AGM of two arbitrary numbers x and y such that $x > y > 0$. Then $M(x, y) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ where a_n and b_n are the arithmetic and geometric means given as:

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}) \qquad (3a)$$

$$b_n = \sqrt{a_{n-1} b_{n-1}} \qquad (3b)$$

In evaluating Equations (3), we note that $a_0 = x$ and $b_0 = y$, whereas $n \geq 1$. An attractive feature of the AGM is that it has a quadratic convergence (see Carvalhaes and Suppes (2008) for proof) and gives very accurate results for a few terms. Therefore, a large number of terms are not needed to get the required accuracy. Since a simple and accurate expression for $K(m)$ is sought in terms of elementary functions, the target is to derive a fourth-term AGM approximation based on four iterations of Equations (3) i.e. $n = 4$. Although it is possible to achieve a better accuracy of the AGM approximation using five iterations and above, the resulting expressions would be too lengthy and difficult to implement using a pocket calculator. Moreover, the improvement in accuracy would be insignificant for most cases.

The recurrence sequences in Equations (3) express the current approximation in terms of the immediate preceding terms of the sequences. Carvalhaes and Suppes (2008) applied this form the AGM sequence to derive a fourth-term approximation that produced a lengthy expression (see Equation (A.4) in appendix) that is difficult to evaluate using a pocket calculator. The idea of the present approach in applying the AGM sequence is to formulate a recurrence relationship that expresses the current approximation in terms of the starting values i.e. a_0 and b_0. From Equations (3), it follows that:

$$a_{n-1} = \frac{1}{2}(a_{n-2} + b_{n-2}) \quad (4a)$$

$$b_{n-1} = \sqrt{a_{n-2}b_{n-2}} \quad (4b)$$

Substituting Equations (4) in (3) and simplifying gives:

$$a_n = \left(\frac{\sqrt{a_{n-2}} + \sqrt{b_{n-2}}}{2}\right)^2 \quad (5a)$$

and

$$b_n = \left(\frac{a_{n-2} + b_{n-2}}{2}\right)^{1/2} (a_{n-2}b_{n-2})^{1/4} \quad (5b)$$

From Equations (5), the current approximation can be obtained from two iterations before. Hence, the second-term approximation can be obtained directly from the initial values. Again, from Equations (5),

$$a_{n-2} = \left(\frac{\sqrt{a_{n-4}} + \sqrt{b_{n-4}}}{2}\right)^2 \quad (6a)$$

and

$$b_{n-2} = \left(\frac{a_{n-4} + b_{n-4}}{2}\right)^{1/2} (a_{n-4}b_{n-4})^{1/4} \quad (6b)$$

Substituting Equations (6) into (5a) and after algebraic simplification we get:

$$a_n = \frac{1}{16}\left[\sqrt{a_{n-4}} + \sqrt{b_{n-4}} + 2\left(\frac{a_{n-4} + b_{n-4}}{2}\right)^{1/4} (a_{n-4}b_{n-4})^{1/8}\right]^2 \quad (7)$$

Using Equation (7), the fourth-term approximation can be obtained directly from the initial values by substituting $n = 4$. Hence,

$$a_4 = \frac{1}{16}\left[\sqrt{a_0} + \sqrt{b_0} + 2\left(\frac{a_0 + b_0}{2}\right)^{1/4} (a_0 b_0)^{1/8}\right]^2 \quad (8)$$

Since the target of a fourth-term approximation is now achieved, the following applies:

$$M(a_0, b_0) \cong a_4 = \frac{1}{16}\left[\sqrt{a_0} + \sqrt{b_0} + 2\left(\frac{a_0 + b_0}{2}\right)^{1/4} (a_0 b_0)^{1/8}\right]^2 \quad (9)$$

Assuming $a_0 = 1$ and $b_0 = \sqrt{1-m} = k'$ where k' is the complementary elliptic modulus, then the approximate expression for the elliptic integral of the first kind can be derived from Equation (9) as:

$$K(m) \cong \frac{8\pi}{\left[1 + (1-m)^{1/4} + 2\left(\frac{1+\sqrt{1-m}}{2}\right)^{1/4} (1-m)^{1/16}\right]^2} \quad (10)$$

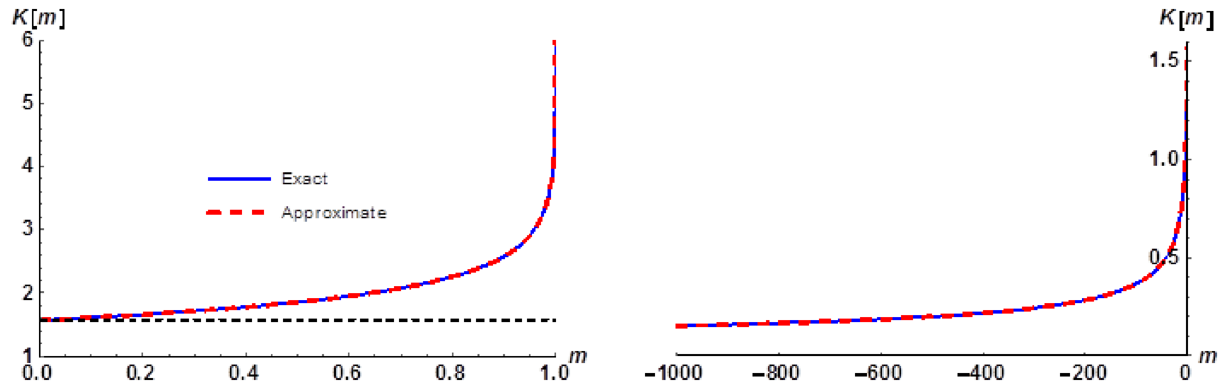

Fig. 1: Exact and approximate computation of the $K(m)$ function for $0 \le m \le 0.9999$ (left) and $-10^3 \le m \le 0$ (right). For coloured plot refer to online version.

Equation (10) can be compared with Equation (A.4) in the appendix to appreciate the simplicity of the present approximation of the $K(m)$ function. Fig. 1 shows a comparison of Equation (10) and the exact $K(m)$ solution for $-10^3 \le m < 1.0$. An excellent match is observed confirming the accuracy of the present approximation. Note that the black horizontal line on the left plot represents the $K(0)$ value of $\pi/2$. The computed results revealed that Equation (10) gives a maximum error of 7.56494×10^{-4} % for $0 \le m \le 0.9999$ and 7.56577×10^{-4} % for $-10^4 \le m \le 0$.

The error analysis confirms the remarkable accuracy of Equation (10) for a wide range of m-values. Hence, it is applied in the next section to derive approximate periodic solutions for pendulum-like motion and Duffing-type oscillators.

3. Periodic solution of pendulum-like and Duffing-type oscillators

3.1 Pendulum-like oscillators

The dynamic equation for the free undamped motion of a pendulum-like system can be expressed as:

$$\ddot{\varphi} + \omega_0^2 \sin\varphi = 0 \tag{11}$$

where ω_0 is based on the system's properties as shown in Table 1. The initial conditions to equation (11) are $\varphi(0) = A$ and $\dot{\varphi}(0) = 0$ where $A \in [0, \pi]$ is the amplitude of the pendulum oscillations in radians. Table 1 shows the expression for ω_0 for various pendulum-like oscillators including the notable simple pendulum.

$$\frac{T_{ex}}{T_0} = \frac{2}{\pi} K(m) \tag{12}$$

where $m = \sin^2(A/2)$ and $T_0 = 2\pi/\omega_0$. Then, substituting this expression for m and Equation (10) into Equation (12), and after simplifying yields the approximate period, T_{app}, as:

Table 1: Expression for ω_0 for some examples of pendulum-like oscillators

S/n	Pendulum-like oscillator	ω_0	Description of constants
1	Simple pendulum (Big-Alabo, 2020a)	$\sqrt{g/l}$	g = gravitational acceleration l = length of the pendulum
2	Compound pendulum (Hinrichsen, 1981)	$\sqrt{\frac{mgL}{I_{cm} + mL^2}}$	m =mass of the body g = gravitational acceleration L = distance between centre of mass and pivot I_{cm} = moment of inertia about centre of mass
3	Eccentrically-loaded hoop (Cross, 2020)	$\sqrt{\frac{MD(g + R\omega^2)}{I_{cm} + MR^2}}$ where $D \ll R$	M = total mass D = distance between centre of hoop and centre of mass g = gravitational acceleration R = radius of hoop I_{cm} = moment of inertia about centre of mass
4	Gravitational lensing in general relativity (Farley, 2018)	$\sqrt{2\Psi}$	Ψ = Weyl driving term
5	Rigid rod sliding on the smooth internal walls of a hollow cylinder (Nayfeh and Mook, 1995)	$\sqrt{\frac{g(R^2 - l^2)^{1/2}}{R^2 - 2l^2/3}}$	g = gravitational acceleration R = radius of cylinder l = half rod length
6	Small solid cylinder rolling without slip on the smooth internal surface of a hollow cylinder (Nayfeh and Mook, 1995)	$\sqrt{\frac{2g}{3(R - r)}}$	g = gravitational acceleration R = radius of large cylinder r = radius of small cylinder

$$\frac{T_{app}}{T_0} = \frac{16}{\left(1 + \sqrt{\cos\left(\frac{A}{2}\right)} + 2\sqrt{\cos\left(\frac{A}{4}\right)\left[\cos\left(\frac{A}{2}\right)\right]^{1/4}}\right)^2} \tag{13}$$

In simplifying T_{app} to get Equation (13), the following trigonometric identities were used: $1 - \sin^2\theta = \cos^2\theta$ and $\cos^2\theta = (1 + \cos 2\theta)/2$. Equation (13) gives a simple explicit formula for the period of the simple pendulum in terms of the amplitude. Such simple explicit period-amplitude formulae are desirable for undergraduate courses (Big-Alabo, 2020a). The accuracy of the present pendulum formula was demonstrated in Table 2, which shows a comparison of the present formula (Equation (13)), the exact formula (Equation (12)) and the approximate formula derived in Big-Alabo (2020a). The present formula produces a maximum relative error of 0.00000% for $A \leq 175°$, 0.00103% for $A \leq 179°$, 0.03428% for $A \leq 179.9°$ and 0.25485% for $A \leq 179.99°$. These results are several orders more accurate than corresponding results obtained by using the formula in Big-Alabo (2020a), except for $A = 179.99°$ where the present formula is about twice more accurate.

In the cases of $A = 179.9°$ and $A = 179.99°$, the respective values of the elliptic modulus are $m = 0.99999924$ and $m = 0.9999999924$. The corresponding relative errors in using Equation (10) to compute $K(m)$ are 0.03423% and 0.2549% respectively because of how close the m-values are to the asymptotic limit of $m = 1.0$. It was observed that these errors are the same errors found in the present pendulum formula for the amplitudes under consideration. The accuracy of the present pendulum formula surpasses all other explicit formulae reported in the literature and is only matched exactly by the formula by Carvalhaes and Suppes (2008). However, the present formula is algebraically simpler compared to the formula by Carvalhaes and Suppes (2008). Hence, while the latter is more suitable for spreadsheet implementation, the present formula can be easily computed with a pocket calculator, making it ideal for undergraduate students.

Table 2: Results of exact and approximate time period formulae for pendulum-like oscillations

		Big-Alabo (2020a)		**This study**	
A **(deg)**	$\boldsymbol{T_{ex}/T_0}$	$\boldsymbol{T_{app}/T_0}$	**% Error**	$\boldsymbol{T_{app}/T_0}$	**% Error**
1	1.00002	1.00002	0.00000	1.00002	0.00000
5	1.00048	1.00048	0.00000	1.00048	0.00000
10	1.00191	1.00191	0.00000	1.00191	0.00000
20	1.00767	1.00767	0.00000	1.00767	0.00000
30	1.01741	1.01741	0.00000	1.01741	0.00000
60	1.07318	1.07318	0.00000	1.07318	0.00000
90	1.18034	1.18031	0.00254	1.18034	0.00000
120	1.37288	1.37262	0.01894	1.37288	0.00000
150	1.76220	1.76052	0.09534	1.76220	0.00000
170	2.43935	2.43270	0.27302	2.43936	0.00000
175	2.87766	2.86737	0.35758	2.87766	0.00000
179	3.90107	3.88464	0.42117	3.90103	0.00103
179.9	5.36687	5.34567	0.39502	5.36503	0.03428
179.99	6.83274	6.79576	0.54120	6.81532	0.25495

3.2 Duffing-type oscillators

Here we consider two types of Duffing oscillators whose exact period are known and expressed in terms of the $K(m)$ function. The first is the classic cubic-Duffing oscillator with a cubic nonlinear stiffness term while the second is the cubic-quintic Duffing oscillator with cubic and quintic nonlinear stiffness terms. The dynamic equations for these oscillators are as follows:

Cubic-Duffing oscillator

$$\ddot{u} + k_1 u + k_3 u^3 = 0 \tag{14}$$

Cubic-quintic Duffing oscillator

$$\ddot{u} + k_1 u + k_3 u^3 + k_5 u^5 = 0 \tag{15}$$

The exact period of the oscillators in Equations (14) and (15) can be expressed as (Big-Alabo, 2020a; Belendez et al., 2016):

$$T_{ex} = 4K(m)/\psi \tag{16}$$

where $m = \frac{A^2 k_3}{2(k_1 + A^2 k_3)}$ and $\psi = \sqrt{k_1 + A^2 k_3}$ for the cubic-Duffing oscillator; whereas $m = \frac{1}{2} - \frac{1}{4} q_3 \sqrt{\frac{3}{2 q_1 q_2}}$ and $\psi = (q_1 q_2 / 6)^{1/4}$ for the cubic-quintic Duffing oscillator. The q_i constants (for $i = 1,2,3$) are expressed in terms of the stiffness constants as follows: $q_1 = k_1 + k_3 A^2 + k_5 A^4$, $q_2 = 6k_1 + 3k_3 A^2 + 2k_5 A^4$ and $q_3 = 4k_1 + 3k_3 A^2 + 2k_5 A^4$. Substituting Equation (10) into (16) gives the approximate period as:

$$T_{app} = \frac{32\pi}{\psi \left[1 + p^{1/4} + 2\left(\frac{1+\sqrt{p}}{2}\right)^{1/4} p^{1/16}\right]^2} \tag{17}$$

where $p = 1 - m$ is the square of the complementary elliptic modulus.

For the cubic-Duffing oscillator, the approximate period can be written as:

$$T_{app} = \frac{32\pi}{\sqrt{k_1 + A^2 k_3} \left[1 + p^{1/4} + 2\left(\frac{1+\sqrt{p}}{2}\right)^{1/4} p^{1/16}\right]^2} \tag{18}$$

Table 3: Exact and approximate results for cubic-Duffing oscillator with double-well potential

k_1	k_3	A	T_{ex}	T_{app}	% Error
– 1.0	1.0	5	1.5286	1.5286	2.905×10^{-14}
– 1.0	1.0	10^1	0.747096	0.747096	0.000
– 1.0	1.0	10^3	7.4163×10^{-3}	7.4163×10^{-3}	1.170×10^{-14}
– 1.0	1.0	10^5	7.4163×10^{-5}	7.4163×10^{-5}	1.827×10^{-14}
– 1.0	1.0	10^9	7.4163×10^{-9}	7.4163×10^{-9}	0.000
– 1.0	10.0	5	0.47042	0.47042	1.180×10^{-14}
– 1.0	10.0	10^1	0.23470	0.23470	1.823×10^{-14}
– 1.0	10.0	10^3	2.3452×10^{-3}	2.3452×10^{-3}	0.000
– 1.0	10.0	10^5	2.3452×10^{-5}	2.3452×10^{-5}	2.889×10^{-14}
– 1.0	10.0	10^9	2.3452×10^{-9}	2.3452×10^{-9}	1.764×10^{-14}
– 10.0	1.0	5	2.41725	2.41725	1.837×10^{-13}
– 10.0	1.0	10^1	0.80290	0.80290	1.383×10^{-14}
– 10.0	1.0	10^3	7.4164×10^{-3}	7.4164×10^{-3}	1.170×10^{-14}
– 10.0	1.0	10^5	7.4163×10^{-5}	7.4163×10^{-5}	0.000
– 10.0	1.0	10^9	7.4163×10^{-9}	7.4163×10^{-9}	0.000

where $p = \frac{2k_1 + A^2 k_3}{2(k_1 + A^2 k_3)}$. For $k_1 > 0$ and $k_3 > 0$, we have a hardening spring and the following is true: $0 < m \leq 1/2$. This implies that the maximum relative error in using Equation (18) is 1.1976×10^{-14} % , which is within machine epsilon precision (i.e. an absolute error less than or equal to $2^{-52} = 2.22 \times 10^{-16}$ based on IEEE floating-point standard). This accuracy is independent of the magnitude of A and is applicable for $0 < A < \infty$. Assuming that $k_3 < 0$, we have a softening spring and $m \leq 1/2$. Depending on the magnitudes of k_1 and k_3, it is possible to have $m < 0$. However, as demonstrated in Section 2, the maximum relative error would still be negligible. Therefore, Equation (18) is accurate for both hardening and softening cubic nonlinear springs.

On the other hand, the approximate period of the cubic-quintic Duffing oscillator cannot be simplified further and is evaluated using Equation (17) with $p = \frac{1}{2} + \frac{1}{4} q_3 \sqrt{\frac{3}{2 q_1 q_2}}$. If $k_1 > 0$, $k_3 > 0$ and $k_5 > 0$, then $0 < m \leq 1/2$ and the accuracy of using Equation (17) is within machine epsilon precision for $0 < A < \infty$. If $k_3 < 0$ or $k_5 < 0$, then we may have a softening or hardening spring depending on the magnitudes of k_3 and k_5. It also implies that $m \leq 1/2$ and we can have $m < 0$

depending on the signs and magnitudes of k_3 and k_5. In any case, the error in the approximate period is still negligible.

One situation in which it is difficult to obtain an accurate estimate of the time period using approximate analytical methods is the cubic-Duffing oscillator with double-well potential. In this case, the oscillator is characterized by a bi-stable equilibrium. This condition occurs when $k_1 < 0$ and $k_3 > 0$ and the points of stability of the system are $u = \sqrt{-k_1/k_3}$ and $u = -\sqrt{-k_1/k_3}$. Also, for real periodic solutions, $p > 0$ which implies that $A > \sqrt{-2k_1/k_3}$. In order to demonstrate the accuracy of Equation (18) to estimate the period of the cubic-Duffing oscillator with double-well potential, we consider typical cases where $k_1 < 0$ and $k_3 > 0$. Table 3 shows the calculations of the approximate and exact period for the typical cases. It was observed that the maximum relative error for the cases considered was less than 1.0×10^{-12} %. Again, this error is close to machine epsilon precision.

4. Conclusion

An accurate analytical solution that is based on elementary functions has been formulated to compute the complete elliptic integral of the first kind. The analytical solution was based on the fourth-term AGM approximation of the $K(m)$ function. The analytical solution of the $K(m)$ function was then applied to derived approximate solutions for the time period of pendulum-like and Duffing-type oscillators; two of the most common categories of nonlinear oscillators that account for the dynamics of several physical systems.

Computational experiments show that the present approximation of the $K(m)$ function gives results that are within machine epsilon precision for most cases, and in extreme cases, it gives results that are well below the acceptable error limit in science and engineering. The approximate formula derived for the period of pendulum-like oscillators showed remarkable accuracy that cannot be matched by other published formulae except the formula of Carvalhaes and Suppes (2008) which matches the accuracy of the present pendulum formula exactly. However, the advantage of the present pendulum formula over the formula of Carvalhaes and Suppes (2008) is that the latter formula (see Equation (A.8) in appendix) is too lengthy and is better suited for spreadsheet computation, while the present one is simple and compact, and can be computed with a pocket calculator. This makes the present formula attractive for classroom application. Also, it has been demonstrated (Carvalhaes and Suppes, 2008) that more accurate pendulum formulae can be obtained by using the fifth- or higher-term approximation of the AGM but the resulting formulae would be too lengthy and not suitable for classroom application.

On the other hand, calculations using the present approximation for the period of the Duffing-type oscillators showed that machine epsilon precision could be achieved for hard and soft Duffing-type oscillators with cubic and cubic-quintic nonlinearities. Investigations on the cubic-Duffing oscillator characterized by a double-well potential showed that the present approximation is very accurate for estimating the period of a double-well cubic-Duffing oscillator.

In conclusion, it is believed that the present study can add pedagogical value in the study of elliptic integrals (i.e. the $K(m)$ function) and periodic oscillations of nonlinear systems. The method for deriving the recursive formula for the AGM is simple and can be easily demonstrated during lectures.

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Appendix

Carvalhaes and Suppes (2008) derived the following approximations of the $K(m)$ function based on the first four-terms of the AGM sequence:

$$K(m) \cong \frac{\pi}{1+\sqrt{1-m}} \qquad (A.1)$$

$$K(m) \cong \frac{2\pi}{1+\sqrt{1-m}+2(1-m)^{1/4}} \qquad (A.2)$$

$$K(m) \cong \frac{4\pi}{1+\sqrt{1-m}+2(1-m)^{1/4}+2^{3/2}(1-m)^{1/8}\left(1+\sqrt{1-m}\right)^{1/2}} \qquad (A.3)$$

$$K(m) \cong \frac{8\pi}{\left\{\begin{matrix} 1+\sqrt{1-m}+2(1-m)^{1/4}+2^{3/2}(1-m)^{1/8}\left(1+\sqrt{1-m}\right)^{1/2}+ \\ 2^{7/4}(1-m)^{1/16}\left(1+\sqrt{1-m}\right)^{1/4}\left(1+\sqrt{1-m}+2(1-m)^{1/4}\right)^{1/2}\end{matrix}\right\}} \qquad (A.4)$$

Substituting $m = \sin^2(A/2)$ in Equations (A.1 – A.4) and simplifying gives the following pendulum period formulae derived by Carvalhaes and Suppes (2008) as:

$$\frac{T_1}{T_0} = \frac{2}{1+\cos\left(\frac{A}{2}\right)} \qquad (A.5)$$

$$\frac{T_2}{T_0} = \frac{4}{1+\cos\left(\frac{A}{2}\right)+2\cos^{1/2}\left(\frac{A}{2}\right)} \qquad (A.6)$$

$$\frac{T_3}{T_0} = \frac{8}{1+\cos\left(\frac{A}{2}\right)+2\cos^{1/2}\left(\frac{A}{2}\right)+2^{3/2}\cos^{1/4}\left(\frac{A}{2}\right)\left[1+\cos\left(\frac{A}{2}\right)\right]^{1/2}} \tag{A.7}$$

$$\frac{T_4}{T_0} = \frac{16}{\left\{\begin{matrix}1+\cos\left(\frac{A}{2}\right)+2\cos^{1/2}\left(\frac{A}{2}\right)+2^{3/2}\cos^{1/4}\left(\frac{A}{2}\right)\left[1+\cos\left(\frac{A}{2}\right)\right]^{1/2}+ \\ 2^{7/4}\cos^{1/8}\left(\frac{A}{2}\right)\left[1+\cos\left(\frac{A}{2}\right)\right]^{1/4}\left[1+\cos\left(\frac{A}{2}\right)+2\cos^{1/2}\left(\frac{A}{2}\right)\right]^{1/2}\end{matrix}\right\}} \tag{A.8}$$

Note that Equation (A.4) gives exactly the same results as Equation (10) and Equation (A.8) gives exactly the same results as Equation (13).